J. Fluid Mech. (1982), vol. 119, pp. 409–422
Printed in Great Britain


# Viscous frictional torque in the flow between two concentric rotating rough cylinders

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(Received 28 July 1981)

This paper describes an experimental study of the effects of surface roughness on the flow between two concentric cylinders, one of which rotates. The surface roughness has some effects on the coefficient of viscous frictional torque $C_M$ in the transient and in the fully developed turbulent region. In the fully developed rough turbulent flow, the value of $C_M$ depends on both the Reynolds number $R_\omega$ and the relative roughness $r_\mathrm{m}/k$ in the case where the outer cylinder rotates, but $C_M$ depends only on $r_\mathrm{m}/k$ in the case where the inner cylinder rotates. The effect of the surface roughness of the inner cylinder is greater than that of the outer one for both cases.

## 1. Introduction

Many studies have been reported on the flow of viscous liquid between two concentric rotating cylinders. Prediction of the onset of instability and information on the laminar and turbulent Taylor vortices are most interesting in the case where the inner cylinder rotates and the outer one is stationary. On the other hand, the critical Reynolds number and the torque transmitted to one cylinder have been considered experimentally in the case where the outer cylinder rotates and the inner one is stationary. Moreover, the unstable-flow phenomenon has been studied in the case where both the cylinders rotate.

The experimental techniques commonly used are measurements of torque or time-dependent velocity component, and flow visualization. The torque measurements have been made over a number of years by various researchers such as Couette (1890), Taylor (1923, 1936), Wendt (1933), Donnelly (1958), Donnelly & Simon (1960), Tillman (1961), Vohr (1968), Yamada, Nakabayashi & Suzuki (1969), Nakabayashi *et al.* (1972) and Nakabayashi, Yamada & Yamada (1977). However, these studies consider only the case of smooth-surfaced cylinders, and no reports dealing with the effects of surface roughness are in evidence.

The present study investigates the effects on the viscous frictional torque coefficient $C_M$ of various surface roughnesses of an inner or outer cylinder when one of the inner and outer cylinders is rotating and the other is stationary, in order to determine whether surface roughness has a role in the development of Taylor instability and turbulence, to ascertain which (inner or outer) cylinder's surface roughness has the greater effect, and to formulate the relationship between $C_M$ and the Reynolds number $R_\omega$ in a turbulent and transition regime.

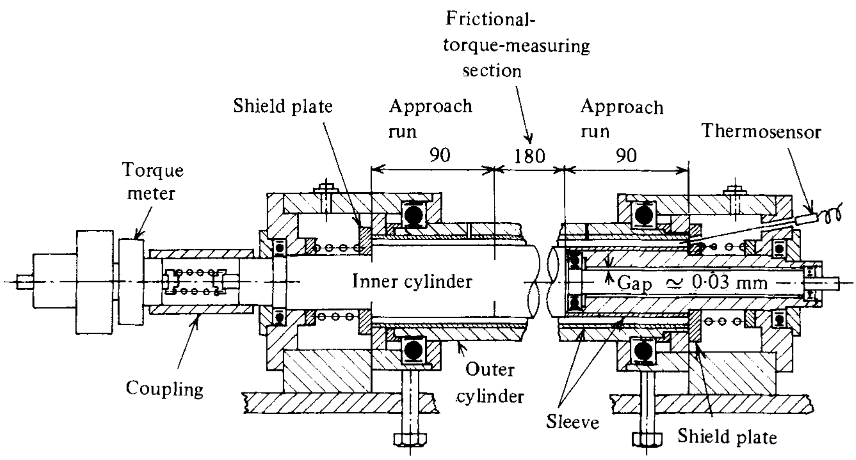



FIGURE 1. Experimental apparatus.

## 2. Experimental apparatus and method

The experimental apparatus is illustrated in figure 1. The inner cylinder with a radius $r_i = 30$ mm was employed. As seen from the figure, it has two approach runs with a length of 90 mm, and a frictional-torque-measuring section 180 mm long. The three components are supported in relation to one another by ball bearings. The gap between the frictional-torque-measuring section and the approach run is extremely small (approx. 0·03 mm), so virtually no water leakage occurs. Moreover, the clearance between the cylinders is shut off at both ends by shield plates, so that there is no effect of secondary flow from the cylinder ends on the measuring section. Thus the torque measurement is expected to be that for cylinders of infinite length. The outer and inner cylinders are maintained in a perfectly concentric condition. Each sleeve is fitted onto the outer cylinder and the shaft, respectively. The diameters are varied in terms of the rough grain diameters so that the clearance $\delta = r_o - r_i$, where $r_o$ is the inner radius of the outer cylinder, will maintain a virtually constant value. The working fluids are freezer oil, glycerol–water solutions of 40 % and 20 % concentrations, and water. A thermosensor was employed to measure fluid temperature when inserted in the clearance between the two cylinders.

As grain specimens, casting sand and abrasive powder were used which had been filtered through a Tyler standard sieve (see table 1). After the grains had bonded to the inner surface of the outer cylinder and the outer surface of the inner cylinder, the respective inner and outer diameters were measured by means of the following volumetric method. When the inner cylinder surface was rough, it was inserted in the smooth-surfaced outer cylinder sleeve so as to be concentric (see figure 2). Then water was poured into the outer cylinder sleeve and inner cylinder clearance. Next, the respective increased and decreased water volumes at the higher and lower water levels were used to calculate the mean inner diameter. Then the outer cylinder's inner diameter was measured with the cylinder gauge. The burette-column water level was measured by means of a reading microscope, the relationship between the burette cross-sectional area and the water height having been obtained previously. The smooth surface of the inner cylinder was also employed to measure the inner

| Symbol | Mean grain diameter $k$ (mm) | Specification |
|---|---|---|
| I | 0·416 | Sand grains (42–48 mesh) |
| II | 0·211 | Sand grains (80–100 mesh) |
| III | 0·105 | Sand grains (170–200 mesh) |
| IV | 0·043 | Abrasive powder (600 mesh) |

TABLE 1. Particles used

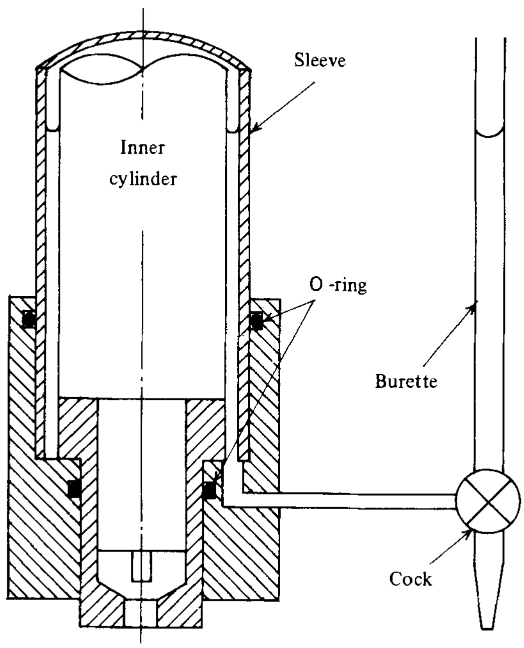



FIGURE 2. Schematic of system for measuring the cylinder diameter.

diameter of the rough-surfaced outer cylinder by the same method. In this way, the volumetric method was used to measure in the case of rough-surface diameters, while the cylinder gauge and micrometer methods were also employed for the smooth-surface measurement. Experiments were conducted for two clearances between the cylinders, namely about 1·5 mm and 2·9 mm. The outer and/or inner diameters of the inner and/or outer cylinders were varied so that the clearance would remain constant in terms of the various grain diameters involved.

Table 2 shows a sample of the results of volumetric measurement when for rough-surface inner cylinder radius $r_1$. The pre-bonding radii make for a consistent inner cylinder radius following grain adhesion in terms of the variation in grain diameters. The adhering grains have a great effect on the accuracy of the experiment. Thus, both before and immediately following the experiment, a micrometer was used to check the rough-surface inner cylinder diameter, as seen in table 2. The deviation when measuring the clearance with the above volumetric method was found to be less than 1 % from a comparison with values obtained with a cylinder gauge and a micrometer

| Symbol | Average radius of smooth-surfaced inner cylinder, $r_i$ (mm) | Average radius of rough-surfaced inner cylinder, before torque measurement, $r_i$ (mm) (by volume of water) | Average radius of rough-surfaced inner cylinder, before torque measurement, $r_i$ (mm) (by micrometer) | Average radius of rough-surfaced inner cylinder, after torque measurement, $r_i$ (mm) (by micrometer) |
|---|---|---|---|---|
| I | 30·44 | 30·70 | 30·91 | 30·90 |
| II | 30·49 | 30·62 | 30·70 | 30·70 |
| III | 30·55 | 30·66 | 30·70 | 30·70 |
| IV | 30·57 | 30·61 | 30·63 | 30·63 |

TABLE 2. Sample measurements of the average radius of the inner cylinder

for both the outer and inner smooth-surface cylinders. Moreover, the difference in the water level when measured by the rise-and-fall methods was 2·5 % or less of the clearance in relation to the rough-surface inner shaft.

## 3. With the outer cylinder rotating

Figure 3 (*a–e*) gives the relationship between the frictional torque coefficient $C_M$ and the rotating Reynolds number $R_\omega$.

$C_M$ and $R_\omega$ are defined as follows:

$$C_M = \frac{M}{2\pi\rho r_i^2 r_o^2 \omega^2}, \quad R_\omega = \frac{r_m \omega \delta}{\nu},$$

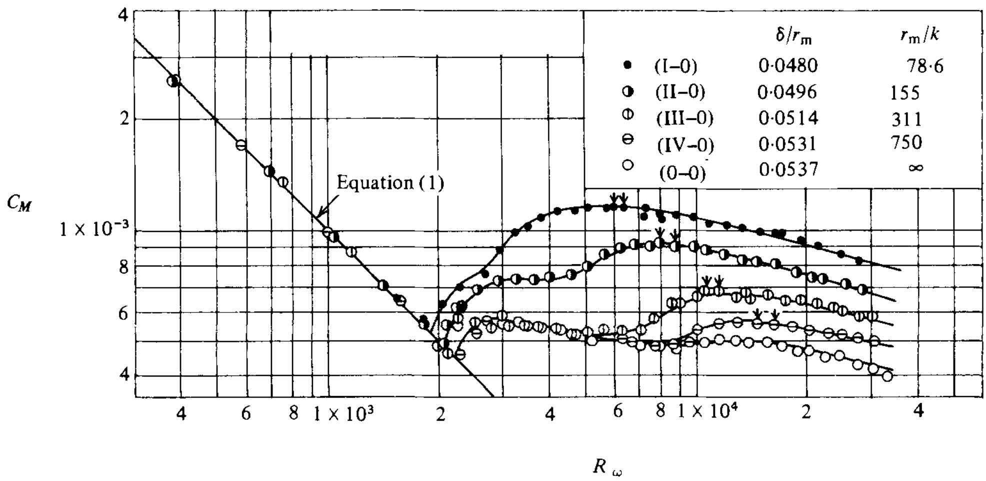
where $M$ is the torque transmitted by fluid friction to the inner cylinder per unit length, $\omega$ is the angular velocity of the inner or the outer cylinder, $\rho$ is the density, $r_m = \frac{1}{2}(r_i + r_o)$ is the mean radius, $\delta = r_o - r_i$ is the clearance, and $\nu$ is the kinematic viscosity of the fluid. Figures 3 (*a, b*) show the results when the inner cylinder has a rough surface and the outer cylinder has a smooth one; figures 3 (*c, d*) show the case of a smooth inner and a rough outer cylinder, and figure 3 (*e*) indicates the results when both surfaces are rough. The numbers in parentheses in the figures represent roughnesses (codes) for the respective inner and outer cylinder (see table 1). Zeros indicate a smooth surface ($k = 0$).

In a laminar flow, the following well-known relationship holds for $C_M$ and $R_\omega$:

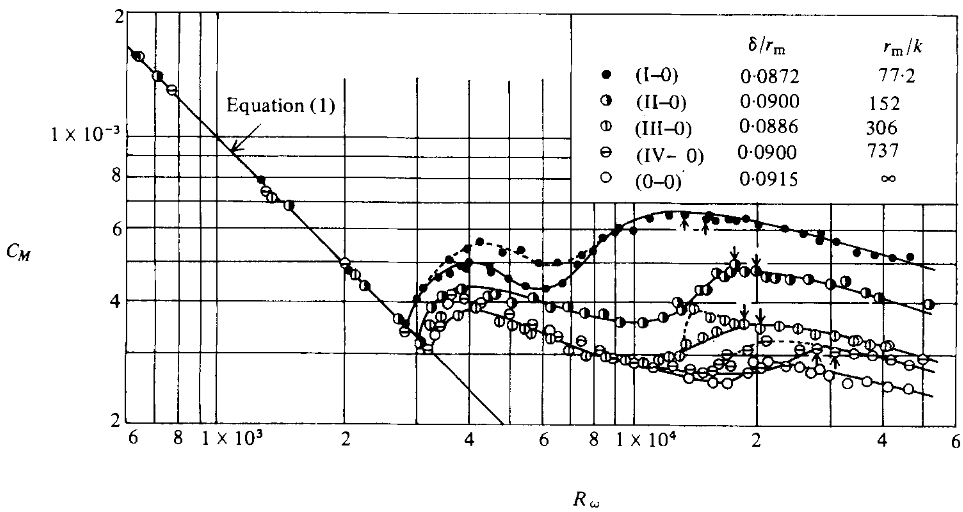
$$C_M = R_\omega^{-1}. \tag{1}$$

The experimental results are all in good agreement with (1) without reference to the relative roughness $r_m/k$, where $k$ is the grain diameter, and it is confirmed that there are no effects of roughness with a laminar flow.

The transition to turbulent flow begins in the vicinity of $R_\omega$ = 2000–3000. Figure 4 presents the relationship between the critical Reynolds number $R_{\omega k}$ at the onset of this transition and the clearance ratio $\delta/r_m$. Although it would appear from figure 3 that there is an $R_{\omega k}$ variation in terms of the $r_m/k$ discrepancy, figure 4 makes it obvious that the discrepancy in $R_{\omega k}$ is rather its cause, since the results of Nakabayashi *et al.* (1977) with a smooth surface $R_{\omega k}$ are shown by a curve. Thus, it is clear that $R_{\omega k}$




(*a*)

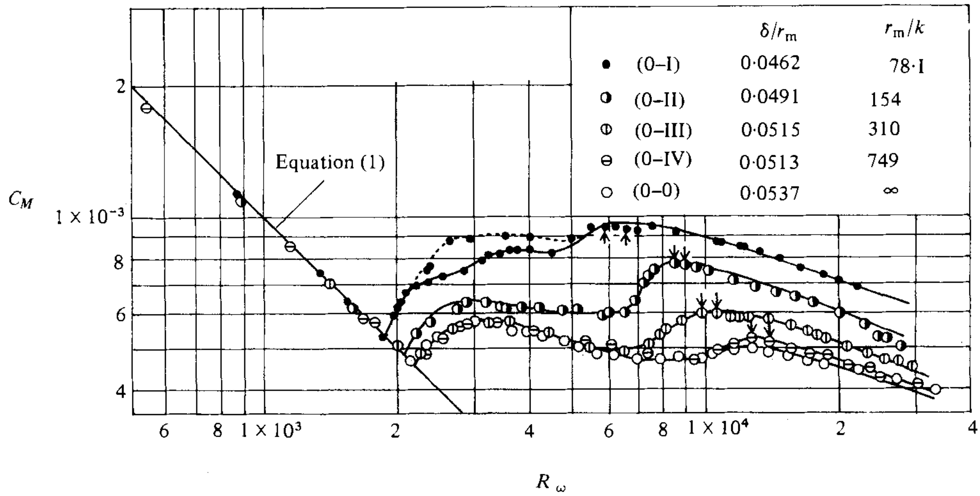



(*b*)

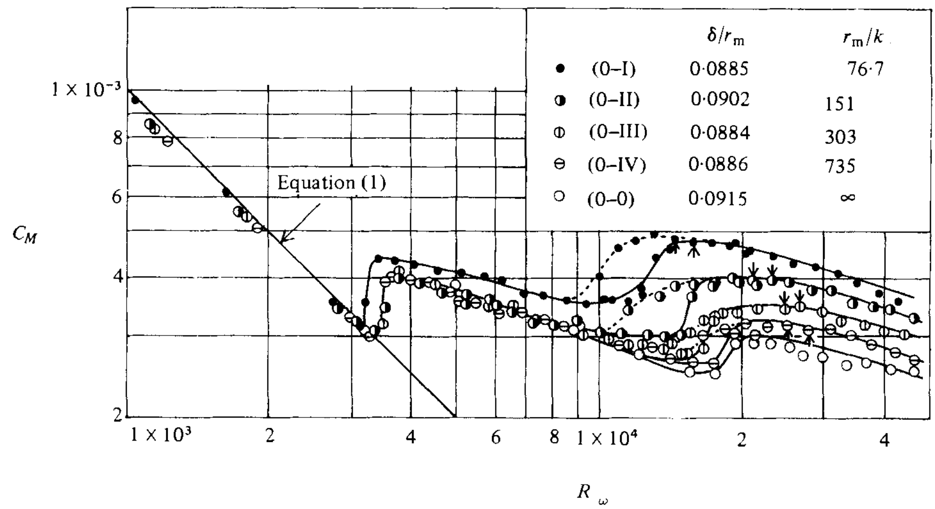
FIGURE 3 (*a*, *b*). Caption on p. 415.

is not influenced by a rough surface. The flow at $R_\omega = 2000\text{–}10^4$ is considered to be in the transition regime, in which $r_m/k = 300$ ($r_m/k = 151$ is included in figure 3(*d*)) fulfils a hydrodynamically smooth surface condition. For $r_m/k$ values less than this, the effect of the surface roughness is remarkable; the value of $C_M$ is especially great as compared with the case for a smooth surface. Again, in the transition regime a hysteresis phenomenon appears in terms of the so-called transition and reverse transition, as shown by the broken and solid lines respectively of figures 3(*b*–*d*). The results indicated by the solid line reflect an increased $R_\omega$, and the broken line shows the results when $R_\omega$ is increased to $2 \times 10^4$ or so, and then are reduced again. This phenomenon was also obtained for the case of both a smooth inner and outer cylinder by Yamada *et al.* (1969).

In the turbulent-flow region ($R_\omega \geqslant 2 \times 10^4$), a completely rough regime can be




(c)




(d)

FIGURE 3 $(c, d)$. Caption on facing page.

obtained in terms of every relative roughness of the grain specimens tested. Just as with the case of a smooth surface, the $C_M$ and $R_\omega$ relationship is given by $C_M \propto R_\omega^{-0.3}$. An instance of this relationship is seen in figure 5, in which the $C_M R_\omega^{0.3}$ and $R_\omega$ relation is indicated when the inner cylinder is rough and the outer one is smooth. The smaller the clearance ratio $\delta/r_{\mathrm{m}}$ and relative roughness $r_{\mathrm{m}}/k$ are, the greater the $C_M R_\omega^{0.3}$ value is. Therefore, in the completely rough turbulent regime, $C_M R_\omega^{0.3}$ is given as a function of $\delta/r_{\mathrm{m}}$ and $r_{\mathrm{m}}/k$, but a difference is also found between the $C_M R_\omega^{0.3}$ values even with different combinations of rough or smooth inner and outer cylinders. Figure 6 shows this rough–smooth combination difference in the relationship between $C_M R_\omega^{0.3}$ and $r_{\mathrm{m}}/k$. From the same figure it is evident that the $C_M R_\omega^{0.3}$ value is greater with a smaller $\delta/r_{\mathrm{m}}$. And for each $\delta/r_{\mathrm{m}}$, it is easy to see that the effects of roughness are far greater with a rough-surfaced inner cylinder and a smooth outer cylinder than with the reverse case. As for the $C_M$ effects with a difference in the rough–smooth combination,

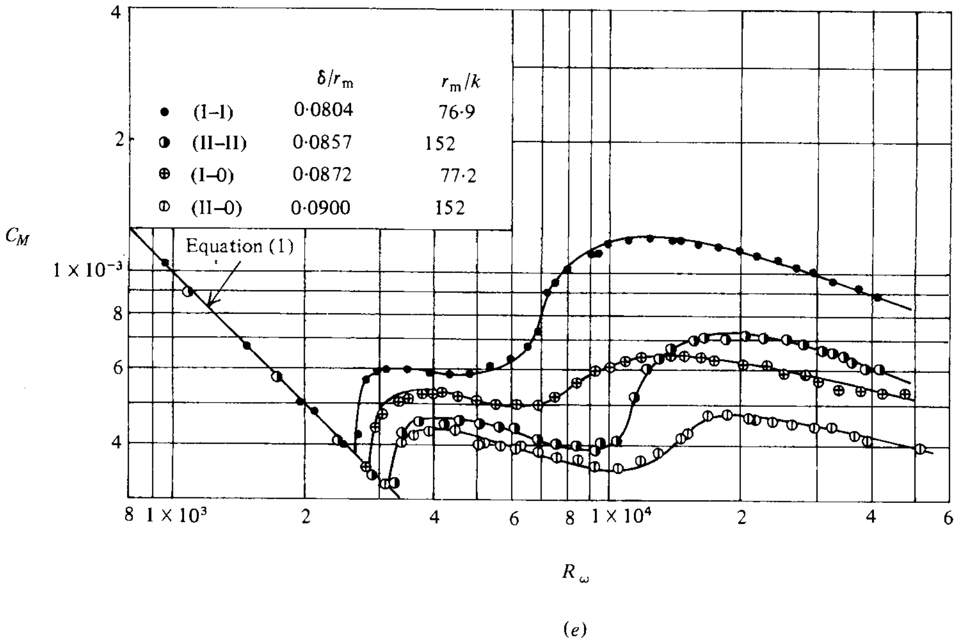



FIGURE 3. Coefficient of frictional torque for a rotating outer cylinder: (*a*) inner rough cylinder, outer smooth cylinder, $\delta/r_m \simeq 0{\cdot}05$; (*b*) inner rough, outer smooth, $\delta/r_m \simeq 0{\cdot}09$; (*c*) inner smooth, outer rough, $\delta/r_m \simeq 0{\cdot}05$; (*d*) inner smooth, outer rough, $\delta/r_m \simeq 0{\cdot}09$; (*e*) inner and outer cylinders both rough, $\delta/r_m \simeq 0{\cdot}08$.

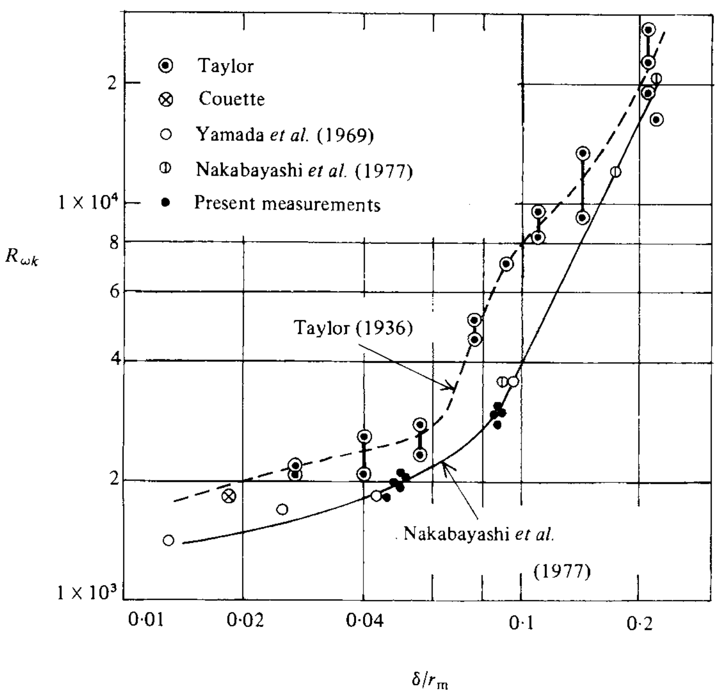



FIGURE 4. Critical Reynolds number for a rotating outer cylinder.

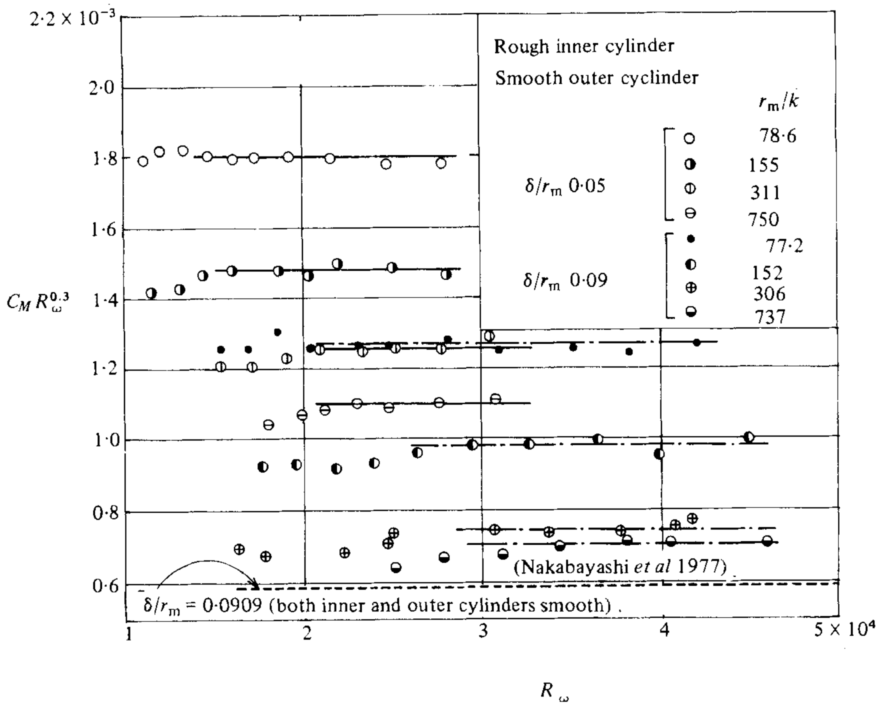



FIGURE 5. Relationship between the coefficient of viscous frictional torque and Reynolds number in the fully developed turbulent flow for a rotating outer cylinder.

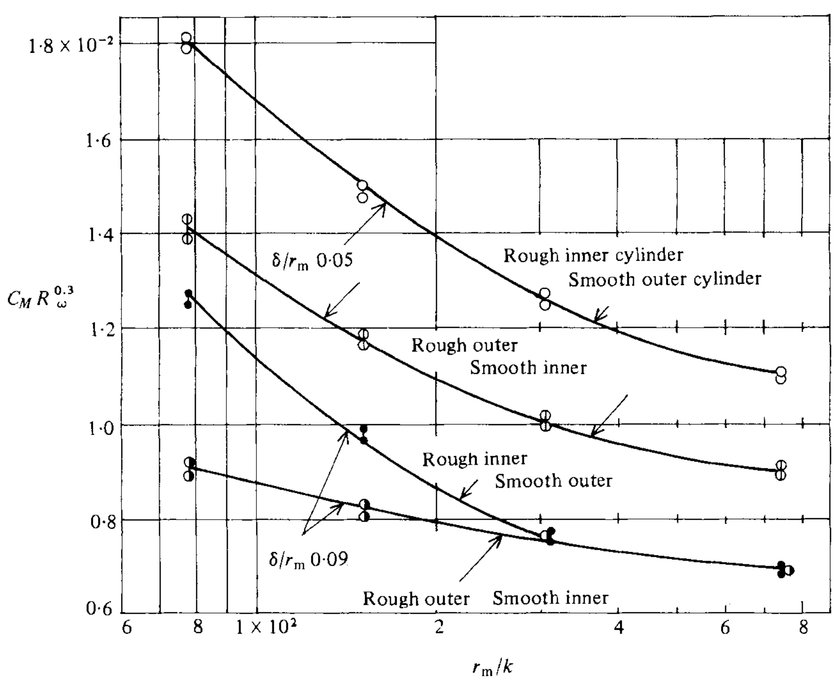



FIGURE 6. Relationship between the coefficient of viscous frictional torque and the relative roughness for a rotating outer cylinder.

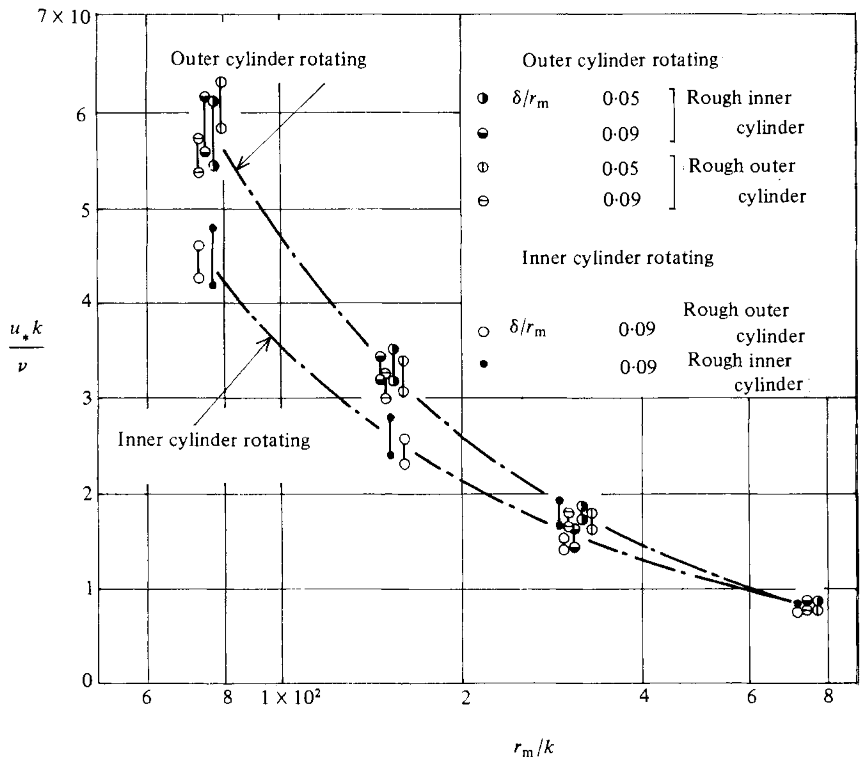
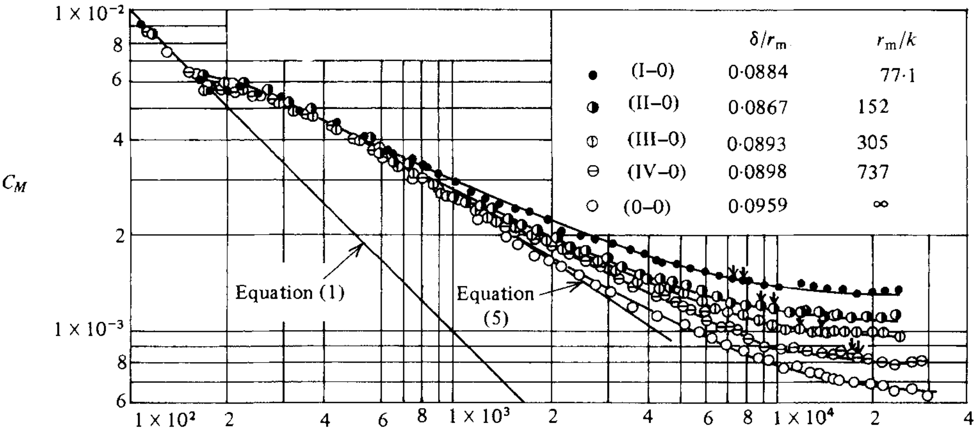



FIGURE 7. Relationship between $u_* k/\nu$ and the relative roughness.

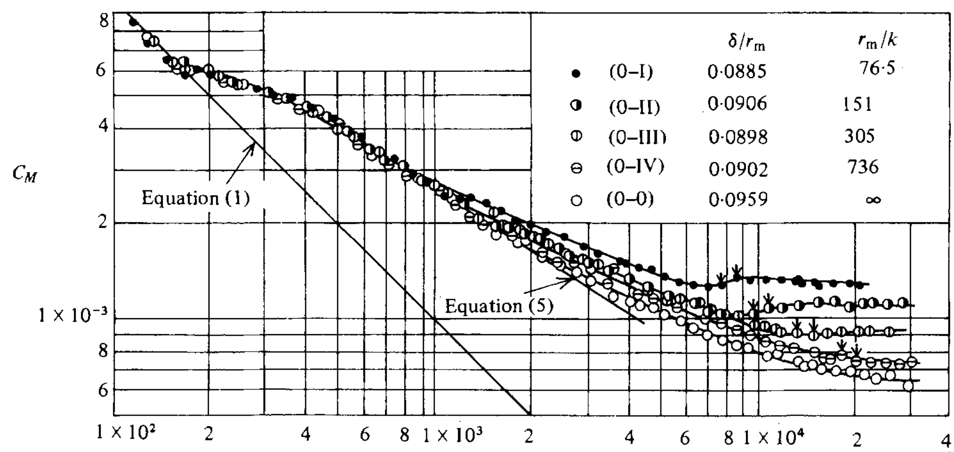
there have been reports on a rotating disk or cone in a vessel that indicate that $C_M$ is greater with a rough-surfaced rotating cylinder than the reverse combination. Yamada & Ito (1976) argued that this is because of the difference in the strength of the secondary flow in such clearance flow. No secondary flow occurs in the flow between two rotating cylinders, so that the $C_M$ difference obtained in the present experimental results with rough–smooth combinations makes it reasonable to assume that this explanation is incorrect. In the case of the rotating inner cylinder, to be dealt with below, the value in the case of a rough-surfaced inner cylinder is slightly greater than the case for a rough-surfaced outer cylinder. Thus, from this standpoint at least it is more likely that friction velocity is a major factor in explaining the above phenomenon. In other words, the cylinder with the greater friction velocity (i.e. the cylinder with a greater shear stress, such as the inner cylinder in the case of two rotating cylinders, or a disk or cone relationship with an experimental system) would have a more marked roughness effect owing to the thinner, viscous sub-layer. Nonetheless, the results of velocity-distribution measurements must be the basis for discussion on this point.

Next, with the above discussion as a basis, we express the threshold value at which the turbulent range begins by means of the Reynolds number $u_* k/\nu$ using the friction velocity $u_*$. Using it for the shear stress on the inner cylinder, we obtain the following equation from the $C_M$ formulation:

$$\frac{u_* k}{\nu} = \left(\frac{\tau i}{\rho}\right)^{\frac{1}{2}} \frac{k}{\nu} = C_M^{\frac{1}{2}} R_\omega \frac{k}{\delta}\left(1 + \frac{1}{2}\frac{\delta}{r_m}\right). \tag{2}$$



(a)



(b)

FIGURE 8. The coefficient of viscous frictional torque for a rotating inner cylinder. (a) Inner rough cylinder and outer smooth cylinder, $\delta/r_m \simeq 0{\cdot}09$; (b) Inner smooth cylinder, outer rough cylinder, $\delta/r_m \simeq 0{\cdot}09$.

With the arrow-indicated experimental point ($R_\omega$ range) in figures 3(*a–d*) as the turbulent-range threshold point, and using (2) with the inner cylinder rough for $\delta/r_m = 0{\cdot}05$ and $0{\cdot}09$, we obtain $u_* k/\nu$ for a rough-surfaced inner and a rough-surfaced outer cylinder, respectively. Figure 7 shows the relation between $u_* k/\nu$ and $r_m/k$. From the same figure, $u_* k/\nu$ is seen to have no relationship either with the smooth–rough combination of the inner/outer cylinder or with $\delta/r_m$, and it decreases together with the increase in $r_m/k$.

When both the inner and outer cylinders have a rough surface (figure 3*e*) the $C_M$ and $R_\omega$ relation tends to be the same as noted above throughout the laminar, transition and turbulent regions. The $C_M$ value in the transition and turbulent regions is much

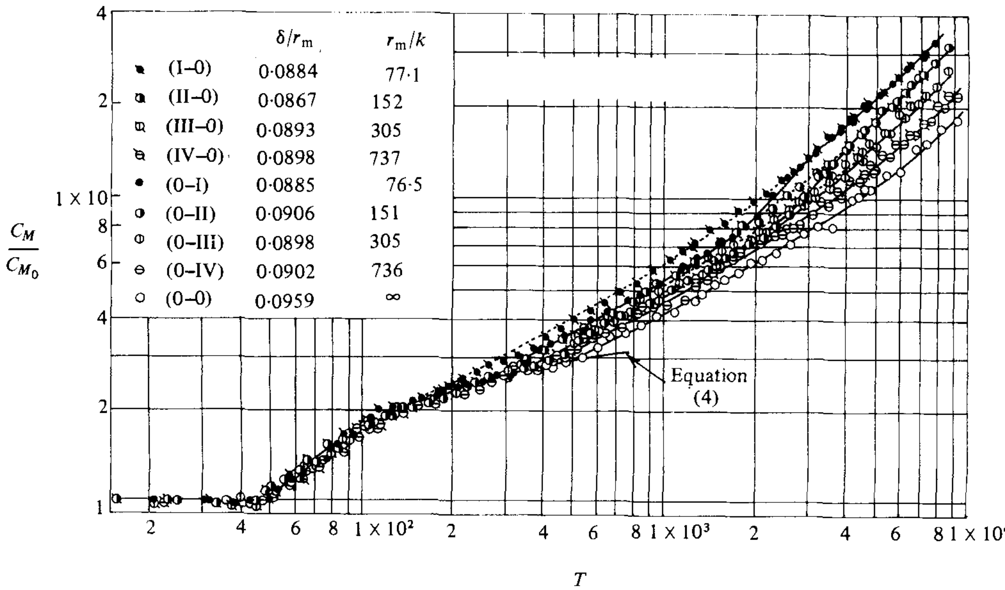



FIGURE 9. Relationship between $C_M/C_{M_0}$ and Taylor number for a rotating inner cylinder.

greater than when one is a rough surface. In the turbulent region ($R_\omega \geqslant 2\times 10^4$) especially, there is 1·6 to 1·7 times the difference when one has a smooth surface.

## 4. With the inner cylinder rotating

Figure 8 gives the relation between the frictional torque coefficient $C_M$ and the rotating Reynolds number $R_\omega$ for a clearance ratio $\delta/r_m$ of approx. 0·09. Figure 8(*a*) gives the case of a rough-surfaced inner cylinder and a smooth-surfaced outer one; figure 8(*b*) shows the reverse combination.

With a laminar flow the relationship between $C_M$ and $R_\omega$ is also given by (1), and the experimental results show complete agreement with (1). At $R_\omega$ of approximately 170, $C_M$ increases from the value given by (1). This is the critical Reynolds number at which Taylor vortices occur, but no difference is noted due to any discrepancy in the relative roughness $r_m/k$.

Next, we consider $C_M$ in the regime in which Taylor vortices develop. Donnelly & Simon (1960) concluded that, for the case in which both the inner and outer cylinders have a smooth surface, the torque $M$ transmitted to the outer cylinder and the angular velocity $\omega$ of the inner cylinder may be expressed in the Taylor-vortex regime by the following relation:

$$M = a\omega^{-1}+b\omega^{1\cdot 36}. \tag{3}$$

However, the constants $a$ and $b$ are functions of the clearance ratio $\delta/r_m$. Taylor (1936), Wendt (1933) and Donnelly (1958) conducted experiments with $\delta/r_m$ of 0·0275 or 0·0403, 0·00668 or 0·381, and 0·0513 or 0·667, respectively. The present authors, however, found no $\delta/r_m \simeq 0\cdot 09$ case. In the present measurements, the $\omega$-power series was also in agreement with (3).

Figure 9 shows a comparison of the frictional torque coefficient $C_M$ and $C_{M_0}$ in

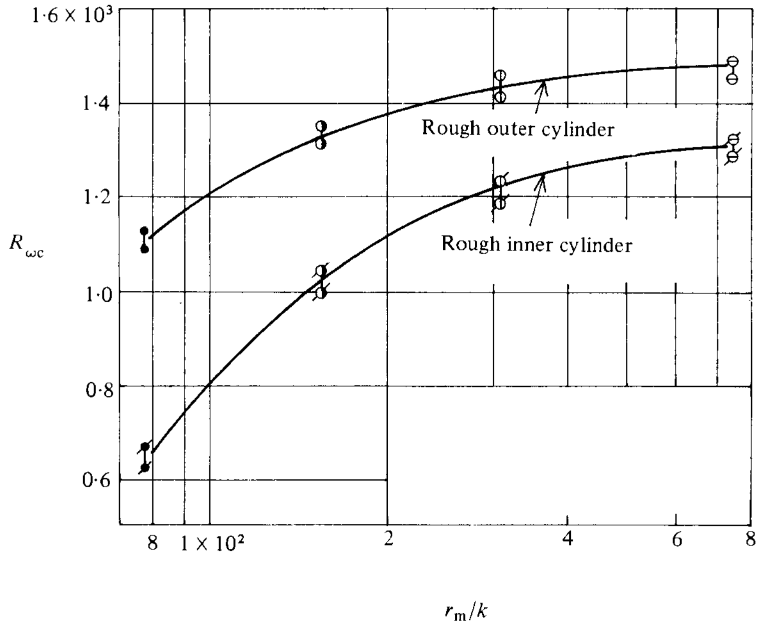



FIGURE 10. Relationship between critical Reynolds number and the relative roughness for a rotating inner cylinder, $\delta/r_m \simeq 0{\cdot}09$.

laminar flow, with $C_M/C_{M_0} = C_M R_\omega$ at $T = R_\omega(\delta/r_m)^{\frac{1}{2}}$. When $T = 46$–$50$, the value of $C_M/C_{M_0}$ increases sharply from 1. In the present experiments on a smooth surface in the range of $\delta/r_m$, the theoretical critical Taylor number was approximately 43 (Chandrasekhar 1958), and the experimental value from the results of frictional-torque measurement was approximately 45 (Nakabayashi *et al.* 1972). Therefore it is fair to say that there was no effect of surface roughness on the critical Taylor number. As can be seen from figure 9, despite the difference in smooth and rough surfaces, the $C_M/C_{M_0}$ relationship to $T$ falls along a single curve from the critical Taylor number to the Taylor number of about 3 times the critical value, giving the following variation of (3):

$$C_M/C_{M_0} = -800T^{-2} + 0{\cdot}335T^{0{\cdot}36}. \quad (4)$$

For the case of both inner and outer smooth cylinders, it is known that the basic toroidal Taylor vortex develops through a wavy-vortex form into the turbulent Taylor-vortex flow with increasing Taylor number above the critical value. Eagles (1974) showed that the theory of Davey, DiPrima & Stuart (1968) of wavy-vortex flow with four azimuthal waves agreed quite well with the experimental results for the torque over the range of Taylor number between 1·1 and 1·2 times the critical Taylor number for the smooth cylinders. Eagles' result has not been found to be adaptable to the case of a rough-surfaced cylinder, because no consideration of the Taylor vortex has been made for that case. However, it may be concluded from Eagles' result that there are no surface-roughness effects on $C_M$ in the region in which Taylor vortices or the wavy Taylor vortices occur. $C_M$ in this flow regime is obtained from the following equations:

$$C_M = -800(\delta/r_m)^{-1} R_\omega^{-3} + 0{\cdot}335(\delta/r_m)^{0{\cdot}18} R_\omega^{-0{\cdot}64}. \quad (5)$$

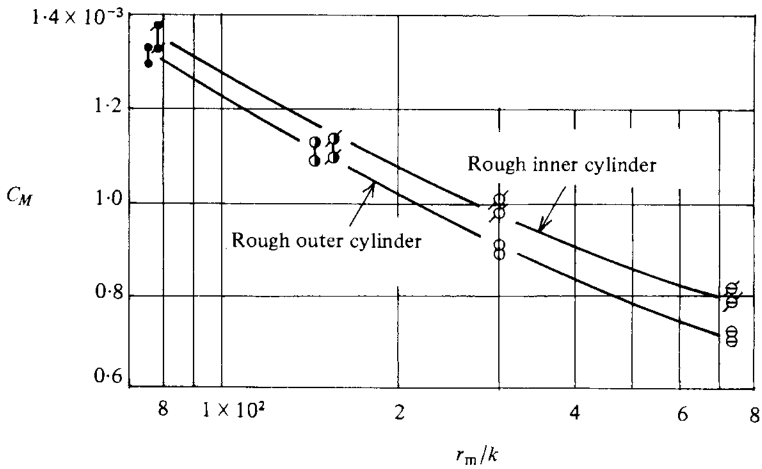



FIGURE 11. The coefficient of turbulent viscous frictional moment for a rotating inner cylinder, $\delta/r_m \simeq 0{\cdot}09$.

In figure 9 the experimental value of $T$ that begins to diverge from (4) is taken for the beginning of the transition from laminar to turbulent flow, and at the same time is considered to be the threshold value $T_c$ at which the effects of roughness become manifest. Let us therefore assume that the threshold Reynolds number at which the surface roughness effects begin to become manifest in $C_M$ can be expressed by $R_{\omega c}$ corresponding to the $T_c$ value.

Figure 10 shows the relationship between this $R_{\omega c}$ and $r_m/k$. $R_{\omega c}$ increases together with $r_m/k$, but that of a rough-surfaced inner cylinder is less than for a rough-surfaced outer cylinder. Turbulent flow is more apt to occur with a rough-surfaced inner cylinder. This tendency also becomes manifest, as seen from what follows. As seen from figure 9, in the $T = T_c \simeq 4000$ region, the $C_M$ value with a rough-surfaced inner cylinder is greater than with a rough-surfaced outer cylinder. Thus, in the above-mentioned transition regime, it is safe to say that the roughness effect of a rough-surfaced inner cylinder is the greater.

Again, in the vicinity of $R_\omega = 2 \times 10^4$ ($T \simeq 6000$) in figure 8, $C_M$ is not dependent on $R_\omega$ but is related only to $r_m/k$. On the other hand, with the flow between two smooth-surfaced cylinders, at $R_\omega = 2 \times 10^4$, $C_M$ decreases as $R_\omega$ increases, so the difference between the cases of rough and smooth surfaces is clear. This flow regime can be considered as a fully developed turbulent and completely rough regime.

Figure 11 gives the relationship between $C_M$ and $r_m/k$; it can be seen that the $C_M$ of an inner cylinder with a rough surface is consistently larger. Since it is like the case of the turbulent flow in figure 8, which begins at the arrow location, the Reynolds number $u_* k/\nu$, using the frictional velocity $u_*$ at this location, may be calculated from (2), just as in §3. As shown in figure 7, $u_* k/\nu$ decreases with the increase in $r_m/k$, but shows no difference with different combinations of rough–smooth surfaces of the inner and outer cylinders. Moreover, compared to the case of a rotating outer cylinder, $u_* k/\nu$ for a rotating inner cylinder is smaller. Hence the surface-roughness effect may be considered to become more easily manifest in a rotating inner cylinder.

## 5. Conclusions

The relationship between the coefficient of viscous frictional torque $C_M$ and Reynolds number $R_\omega$ is considered for two kinds of clearance ratio $\delta/r_m$ and various values of relative roughness $r_m/k$ in the flow between two concentric rough and/or smooth cylinders, one of which rotates. The results obtained are as follows.

(i) The surface roughness of the cylinder has a great effect on $C_M$ in both the transient and the fully developed turbulent flow regions, respectively. The relationship between $C_M$ and $R_\omega$ depends remarkably on $r_m/k$ and combinations of rough and/or smooth cylinders. The surface roughness of the inner cylinder has a greater effect on $C_M$ than that of the outer one.

(ii) For the case of a rotating outer cylinder the experimental values of $C_M R_\omega^{0.3}$ depend on both $r_m/k$ and $\delta/r_m$, and increase with decreasing $r_m/k$ and $\delta/r_m$, in the fully developed turbulent-flow region. Moreover, the experimental results of $C_M$ for both inner and outer rough cylinders are 1·6–1·7 times those for the case with an inner rough cylinder and an outer smooth one.

(iii) For the case of a rotating inner cylinder the surface roughness does not influence Taylor instability. The experimental results for $C_M$ depend only on $r_m/k$ and increase with decreasing $r_m/k$ in the fully developed turbulent-flow region.

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